ARITHMETICES PRINCIPIA

NOVA METHODO EXPOSITA

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REGIS BIBLIOPOLAE

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PRAEFATIO

Quaestiones, quae ad mathematicae fundamenta pertinent, etsi hisce temporibus a multis tractatae, satisfacienti solutione et adhuc carent. Hic difficultas maxime ex sermonis ambiguitate oritur.

Quare summi interest verba ipsa, quibus utimur attente perpendere. Hoc examen mihi proposui, atque mei studii resultatus, et arithmeticae applicationes in hoc scripto expono.

Ideas omnes quae in arithmeticae principiis occurrunt, signis indicavi, ita ut quaelibet propositio his tantum signis enuncietur.

Signa aut ad logicam pertinent, aut proprie ad arithmeticam. Logicae signa quae hic occurrunt, sunt numero ad decem, quamvis non omnia necessaria. Horum signorum usus et proprietates nonnullae in priore parte communi sermone explicantur. Ipsorum theoriam fusius hic exponere nolui. Arithmeticae signa, ubi occurrunt, explicantur.

His notationibus quaelibet propositio formam assumit atque praecisionem, qua in algebra aequationes gaudent, et a propositionibus ita scriptis aliae deducuntur, idque processis qui aequationum resolutioni assimilantur. Hoc caput totius scripti.

Sique, confectis signis quibus arithmeticae propositiones scribere possim, in earum tractatione usus sum methodo, quam quia et in aliis studiis sequenda foret, breviter exponam.

Ex arithmeticae signis quae caeteris, una cum logicae signis exprimere licet, ideas significant quas definire possumus. Ita omnia definivi signa, si quatuor excipias, quae in explicationibus § 1 continentur. Si, ut puto, haec ulterius reduci nequeunt, ideas ipsis expressas, ideis quae prius notae supponuntur, definire non licet.

Propositiones, quae logicae operationibus a caeteris deducuntur, sunt *theoremata;* quae vero non, *axiomata* vocavi. Axiomata hic sunt novem (§ 1), et signorum, quae definitione carent, proprietates fundamentales exprimunt.

In § 1-6 numerorum proprietates communes demonstravi; brevitatis causa, demonstrationes praecedentibus similes omisi; demonstrationum communem formam immutare oportet ut logicae signis exprimantur; haec transformatio interdum difficilior est, tamen inde demonstrationis natura clarissime patet.

In sequentibus § varia tractavi, ut huius methodi potentia magis videatur.

In § 7 nonnulla theoremata, quae ad numerorum theoriam pertinent, continentur. In § 8 et 9 rationalium et irrationalium definitiones inveniuntur.

Denique, in § 10, theoremata exposui nonnulla, quae nova esse puto, ad entium theoriam pertinentia, quae cl.[mus] CANTOR *Punktmenge* (*ensemble de points*) vocavit.

In hoc scripto aliorum studiis usus sum. Logicae notationes et propositiones quae in num. II, III et IV continentur, si nonnullas excipias, ad multorum opera, inter quae BOOLE praecipue, referenda sunt (*).

(*) BOOLE: *The mathematical analysis of logic,* etc. Cambridge, 1847.
— *The calculus of logic.* Camb. and Dublin Math. Journal, 1848.
— *An investigation of the laws of thought,* etc. London, 1854.

E. SCHRÖDER: *Der Operationskreis des Logikkalculs,* Leipzig, 1877.

Ipse iam nonnulla quae ad logicam pertinent tractavit in praecedenti opera.
— *Lehrbuch der Arithmetik und Algebra,* etc. Leipzig, 1873.

Boole e Schröder theorias brevissime exposui in meo libro *Calcolo geometrico* etc. Torino, 1888.

Vide:

C. S. PEIRCE, *On the Algebra of logic;* American Journal, III, 15; VII, 180.

JEVONS. *The principles of science.* London, 1883.

MC.COLL. *The calculus of equivalent statements.* Proceedings of the London Math. Society, 1878. Vol. IX, 9. Vol X, 16.

Signum ϵ, quod cum signo $\ni$ confundere non licet, inversionis in logica applicationes, et paucas alias instĭtui conventiŏnes, ut ad exprimendam quamlibet propositionem pervenirem.

In arithmeticae demonstrationibus usus sum libro: H. GRASSMANN, *Lehrbuch der Arithmetik*, Berlin 1861.

Utilius quoque mihi fuit recens scriptum: R. DEDEKIND, *Was sind und was sollen die Zahlen;* Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur.

Hic meus libellus ut novae methodi specimen habendus est. Hisce notationibus innumeras alias propositiones, ut quae ad rationales et irrationales pertinent, enunciare et demonstrare possumus. Sed, ut aliae theoriae tractentur, nova signa, quae nova indicant entia, instituere necesse est. Puto vero his tantum logicae signis propositiones cuiuslibet scientiae exprimi posse, dummodo adiungantur signa quae entia huius scientiae representant.

SIGNORUM TABULA

Logicae signa

Signum	Significatio	Pag.
P	*propositio*	VII
K	*classis*	X
∩	*et*	VII, X
∪	*vel*	VIII, X, XI
—	*non*	VIII, X
Λ	*absurdum* aut *nihil*	VIII, XI
Ɔ	*deducitur* aut *continetur*	VIII, XI
=	*est aequalis*	VIII
ε	*est*	X
[ ]	inversionis signum	XI
϶	*qui* vel [ε]	XII
Th	*Theorema*	XVI
Hp	*Hypothesis*	»
Ts	*Thesis*	»
L	*Logica*	»

Arithmeticae signa

Signa 1, 2, ..., =, >, <, +, —, × vulgarem habent significationem. Divisionis signum est /.

Signum	Significatio	Pag.
N	*numerus integer positivus*	1
R	*num. rationalis positivus*	12
Q	*quantitas*, sive *numerus realis positivus*	16
Np	*numerus primus*	9
M	*maximus*	6
W	*minimus*	6
T	*terminus*, vel *limes summus*	15
D	*dividit*	9
ɑ	*est multiplex*	9
π	*est primus cum*	9

Signa composita

—<	*non est minor*
=∪>	*est aequalis aut maior*
϶ D	*divisor*
M ϶ D	*maximus divisor*

Logicae notationes.

I. *De punctuatione.*

Litteris a, b,... x, y,... $x'y'$... entia indicamus indeterminata quaecumque. Entia vero determinata signis, sive litteris P, K, N,... indicamus.

Signa plerumque in eadem linea scribemus. Ut ordo pateat quo ea coniungere oporteat, *parenthesibus* ut in algebra, sive *punctis* . : .˙. : : etc. utimur.

Ut formula punctis divisa, intelligatur, primum signa quae nullo puncto separantur colligenda sunt, postea quae uno puncto, deinde quae duobus punctis, etc.

Ex. g. sint a, b, c,... signa quaecumque. Tunc $ab.cd$ significat $(ab)(cd)$; et $ab.cd:ef.gh\therefore k$ significat $(((ab)(cd))((ef)(gh)))k$.

Punctuationis signa omittere licet si formulae quae diversa punctuatione existerent eundem habeant sensum; vel si una tantum formula, et ipsa quam scribere volumus, sensum habeat.

Ut ambiguitatis periculum absit, aritmeticae operationum signis .: nunquam utimur.

Parenthesum figura una est ( ); si in eadem formula, parentheses et puncta occurrant, primum quae parenthesibus continentur, colligantur.

II. *De propositionibus.*

Signo P significatur *propositio.*

Signum $\cap$ legitur *et.* Sint a, b, propositiones; tunc $a \cap b$ est simultanea affirmatio propositionum a, b. Brevitatis causa, loco $a \cap b$ vulgo scribemus $a\,b$.

Signum $-$ legitur *non*. Sit a quaedam P; tunc $-a$ est negatio propositionis a.

Signum $\cup$ legitur *vel*. Sint a, b propositiones; tunc $a \cup b$ idem est ac $-:-a.-b$.

[Signo V significatur *verum*, sive *identitas;* sed hoc signo nunquam utimur].

Signum Λ significat *falsum*, sive *absurdum*.

[Signum C significat *est consequentia;* ita $b \mathrm{C} a$ legitur *b est consequentia propositionis a*. Sed hoc signo nunquam utimur].

Signum $\supset$ significat *deducitur;* ita $a \supset b$ idem significat quod $b \mathrm{C} a$. Si propositiones a, b entia indeterminata continent $x, y, \ldots$, scilicet sunt inter ipsa entia conditiones, tunc $a \supset_{x, y, \ldots} b$ significat: quaecumque sunt $x, y, \ldots$, a propositione a deducitur b. Si vero ambiguitatis periculum absit, loco $\supset_{x, y, \ldots}$ scribemus solum $\supset$.

Signum $=$ significat *est aequalis*. Sint a, b propositiones; tunc $a = b$ idem significat quod $a \supset b . b \supset a$; propositio $a =_{x, y, \ldots} b$ idem significat quod $a \supset_{x, y, \ldots} b . b \supset_{x, y, \ldots} a$.

III. *Logicae propositiones*.

Sint $a, b, c, \ldots$ propositiones. Tunc erit:

1. $a \supset a$.
2. $a \supset b . b \supset c : \supset : a \supset c$.
3. $a = b . = : a \supset b . b \supset a$.
4. $a = a$.
5. $a = b . = . b = a$.
6. $a = b . b \supset c : \supset . a \supset c$.
7. $a \supset b . b = c : \supset . a \supset c$.
8. $a = b . b = c : \supset . a = c$.
9. $a = b . \supset . a \supset b$.
10. $a = b . \supset . b \supset a$.

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11. $ab \supset a$.
12. $ab = ba$.
13. $a(bc) = (ab)c = abc$.

14. $aa = a.$

15. $a = b . \supset . ac = bc.$

16. $a \supset b . \supset . ac \supset bc.$

17. $a \supset b . c \supset d : \supset . ac \supset bd.$

18. $a \supset b . a \supset c : = . a \supset bc.$

19. $a = b . c = d : \supset . ac = bd.$

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20. $-(-a) = a.$

21. $a = b . = . -a = -b.$

22. $a \supset b . = . -b \supset -a.$

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23. $a \cup b . = \therefore - : -a . -b.$

24. $-(ab) = (-a) \cup (-b).$

25. $-(a \cup b) = (-a)(-b).$

26. $a \supset . a \cup b.$

27. $a \cup b = b \cup a.$

28. $a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c.$

29. $a \cup a = a.$

30. $a(b \cup c) = ab \cup ac.$

31. $a = b . \supset . a \cup c = b \cup c.$

32. $a \supset b . \supset . a \cup c \supset b \cup c.$

33. $a \supset b . c \supset d : \supset : a \cup c . \supset . b \cup d.$

34. $b \supset a . c \supset a : = . b \cup c \supset a.$

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35. $a - a = \Lambda.$

36. $a \Lambda = \Lambda.$

37. $a \cup \Lambda = a.$

38. $a \supset \Lambda . = . a = \Lambda.$

39. $a \supset b . = . a - b = \Lambda.$

40. $\Lambda \supset a.$

41. $a \cup b = \Lambda . = : a = \Lambda . b = \Lambda.$

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42. $a \supset . b \supset c : = : ab \supset c.$

43. $a \supset . b = c : = . ab = ac.$

Sit α quoddam relationis signum (ex. gr. $=, \supset$), ita ut $a \, \alpha \, b$ sit quaedam propositio. Tunc loco $- . a \, \alpha \, b$ scribemus $a - \alpha \, b$; scilicet:

$$a - = b . = : - . a = b.$$

$$a - \supset b . = : - . a \supset b.$$

Ita signum $- =$ significat *non est aequalis*. Si propositio a indeterminatum continet x, $a - =_x \Lambda$ significat: sunt x quae conditioni a satisfaciunt. Signum $- \supset$ significat *non deducitur*.

Similiter, si α et β sunt relationis signa, loco $a \, \alpha \, b . a \, \beta \, b$, et $a \, \alpha \, b . \cup . a \, \beta \, b$ scribere possumus $a . \alpha \beta . b$ et $a . \alpha \cup \beta . b$. Ita, si a et b sunt propositiones, formula $a . \supset - = . b$ dicit: ab a deducitur b, sed non vice versa.

$$a . \supset - = . b : = : a \supset b . b - \supset a.$$

Formulae:

$$a \supset b . b \supset c . a - \supset c : = \Lambda.$$

$$a = b . b = c . a - = c : = \Lambda.$$

$$a \supset b . b \supset - = c : \supset . a \supset - = c.$$

$$a \supset - = b . b \supset c : \supset . a \supset - = c.$$

Sed his notationibus raro utimur.

IV. *De classibus.*

Signo K significatur *classis*, sive entium aggregatio.

Signum ϵ significat *est*. Ita $a \epsilon b$ legitur *a est quoddam b*; $a \epsilon \mathrm{K}$ significat *a est quaedam classis*; $a \epsilon \mathrm{P}$ significat *a est quaedam propositio*.

Loco $-(a \epsilon b)$ scribemus $a - \epsilon b$; signum $- \epsilon$ significat *non est*; scilicet:

44. $a - \epsilon b . = : - . a \epsilon b.$

Signum $a, b, c \epsilon m$ significat: a, b et c sunt m; scilicet:

45. $a, b, c \epsilon m . = : a \epsilon m . b \epsilon m . c \epsilon m.$

Sit a classis; tunc $-a$ significatur classis individuis constituta quae non sunt a.

46. $a \epsilon \mathrm{K} . \supset : x \epsilon - a . = . x - \epsilon a.$

Sint a, b classes; $a \cap b$, sive $a b$, est classis individuis constituta quae eodem tempore sunt a et b; $a \cup b$ est classis individuis constituta qui sunt a vel b.

47. $a, b \varepsilon \mathrm{K} . \supset \therefore x \varepsilon . a b :=: x \varepsilon a . x \varepsilon b.$

48. $a, b \varepsilon \mathrm{K} . \supset \therefore x \varepsilon . a \cup b :=: x \varepsilon a . \cup . x \varepsilon b.$

Signum Λ indicat classem quae nullum continet individuum. Ita:

49. $a \varepsilon \mathrm{K} . \supset \therefore a = \Lambda :=: x \varepsilon a . =_x \Lambda.$

[Signo V, quod classem ex omnibus individuis constitutam, de quibus quaestio est, indicat, non utimur].

Signum $\supset$ significat *continetur*. Ita $a \supset b$ significat *classis a continetur in classi b*.

50. $a, b \varepsilon \mathrm{K} . \supset \therefore a \supset b :=: x \varepsilon a . \supset_x . x \varepsilon b.$

[Formula $b \mathrm{C} a$ significare potest *classis b continet classem a;* at signo C non utimur].

Hic signa Λ et $\supset$ significationem habent quae paullo a praecedenti differt; sed nulla orietur ambiguitas. Nam si de propositionibus agatur, haec signa legantur *absurdum* et *deducitur;* si vero de classibus, *nihil* et *continetur.*

Formula $a = b$, si a et b sint classes, significat $a \supset b . b \supset a$. Itaque

51. $a, b \varepsilon \mathrm{K} . \supset \therefore a = b :=: x \varepsilon a . =_x . x \varepsilon b.$

Propositiones 1... 41 quoque subsistunt, si a, b... classes indicant; praeterea est:

52. $a \varepsilon b . \supset . b \varepsilon \mathrm{K}.$

53. $a \varepsilon b . \supset . b - = \Lambda.$

54. $a \varepsilon b . b = c : \supset . a \varepsilon c.$

55. $a \varepsilon b . b \supset c : \supset . a \varepsilon c.$

Sit s classis, et k classis quae in s contineatur; tunc dicimus k esse individuum classis s, si k ex uno tantum constat individuo. Itaque:

56. $s \varepsilon \mathrm{K} . k \supset s : \supset :: k \varepsilon s . = \therefore k - = \Lambda : x, y \varepsilon k . \supset_{x,y} . x = y.$

V. *De inversione.*

Inversionis signum est [ ], eiusque usum in sequenti numero explicabimus. Hic tantum casus particulares exponimus.

1. Sit a propositio, indeterminatum continens x; tunc scriptura $[x] \epsilon\, a$, quae legitur *ea x quibus a*, sive *solutiones*, vel *radices* conditionis a, classem significat individuis constitutam, quae conditioni a satisfaciunt. Itaque:

57. $a \epsilon \mathrm{P} . \supset : [x \epsilon]\, a . \epsilon \mathrm{K}.$

58. $a \epsilon \mathrm{K} . \supset .\!\cdot. [x \epsilon] . x \epsilon a : = a.$

59. $a \epsilon \mathrm{P} . \supset .\!\cdot. x \epsilon . [x \epsilon]\, a : = a.$

Sint α, β, propositiones indeterminatum continentes x; erit:

60. $[x \epsilon] (\alpha \beta) = ([x \epsilon]\, \alpha)([x \epsilon]\, \beta).$

61. $[x \epsilon] - \alpha = - [x \epsilon]\, \alpha.$

62. $[x \epsilon] (\alpha \cup \beta) = [x \epsilon]\, \alpha \cup [x \epsilon]\, \beta.$

63. $\alpha \supset_x \beta . = . [x \epsilon]\, \alpha \supset [x \epsilon]\, \beta.$

64. $\alpha =_x \beta . = . [x \epsilon]\, \alpha = [x \epsilon]\, \beta.$

2. Sint x, y entia quaecumque; systema ex ente x et ex ente y compositum ut novum ens consideramus, et signo (x, y) indicamus; similiterque si entium numerus maior fit. Sit α propositio indeterminata continens x, y; tunc $[(x, y) \epsilon]\, \alpha$ significat classem entibus (x, y) constitutam, quae conditioni α satisfaciunt. Erit:

65. $\alpha \supset_{x, y} \beta . = . [(x, y) \epsilon]\, \alpha \supset [(x, y) \epsilon]\, \beta.$

66. $[(x, y) \epsilon]\, \alpha - = \Lambda . = .\!\cdot. [x \epsilon] . [y \epsilon]\, \alpha - = \Lambda : - = \Lambda.$

3. Sit $x \alpha y$ relatio inter indeterminata x et y (ex. g. in logica relationes $x = y$, $x - = y$, $x \supset y$; in arithmetica $x < y$, $x > y$, etc). Tunc signo $[\epsilon\, \alpha]\, y$ ea x indicamus, quae relationi $x \alpha y$ satisfaciunt. Commoditatis causa, loco $[\epsilon]$, signo ∍ utimur. Ita $\ni \alpha y . = : [x \epsilon] . x \alpha y$, et signum ∍ legitur *qui*, vel *quae*. Ex. gr. sit y numerus; tunc $\ni < y$ classem indicat numeris x compositam qui conditioni $x < y$ satisfaciunt, scilicet, *qui sunt minores* y, vel simpliciter *minores* y. Similiter, quum signum D significet *dividit*, vel *est divisor*, formula ∍D significat *qui dividunt* vel *divisores*. Deducitur $x \epsilon \ni \alpha y = x \alpha y$.

4. Sit α formula indeterminatum continens x. Tunc scriptura $x' [x]\, \alpha$, quae legitur *x' loco* x *in* α *substituto*, formulam indicat quae obtinetur si in α, loco x, x' legimus. Deducitur $x [x]\, \alpha = \alpha$.

5. Sit α formula, quae indeterminata $x, y, ...$ continet. Tunc

$$(x', y', ...) [x, y, ...]\, \alpha,$$

quae legitur $x' y', \ldots$ *loco* x, y,... *in* α *substitutis*, formulam indicat quae obtinetur si in α loco $x, y, \ldots$, litterae $x' y', \ldots$ scribantur. Deducitur $(x, y)\,[x, y]\,\alpha = \alpha$.

VI. *De functionibus.*

Logicae notationes quae praecedunt exprimendae cuilibet arithmeticae propositioni sufficiunt, iisdemque tantum utimur. Hic notationes alias nonnullas breviter explicamus, quae utiles fieri possunt.

Sit s quaedam classis; supponimus aequalitatem inter entia systematis s definitam, quae conditionibus satisfaciat:

$$a = a.$$
$$a = b \,.=.\, b = a.$$
$$a = b \,.\, b = c :\supset .\, a = c.$$

Sit φ signum, sive signorum aggregatus, ita ut si x est ens classis s, scriptura $\varphi\, x$ novum indicet ens; supponimus quoque aequalitatem inter entia $\varphi\, x$ definitam; et si x et y sunt entia classis s, et est $x = y$, supponimus deduci posse $\varphi\, x = \varphi\, y$. Tunc signum φ dicitur esse *functionis praesignum in classi* s, et scribemus $\varphi \ni \mathrm{F}' s$.

$$s \,\epsilon\, \mathrm{K} \,.\supset ::\, \varphi \,\epsilon\, \mathrm{F}' s \,.=\therefore\, x, y \,\epsilon\, s \,.\, x = y :\supset_{x, y} .\, \varphi\, x = \varphi\, y.$$

Verum si, cum sit x quodlibet ens classis s, scriptura $x\varphi$ novum indicet ens, et, ex $x = y$ deducitur $x\varphi = y\varphi$, tunc dicimus φ esse *functionis postsignum in classi* s et scribemus $\varphi \,\epsilon\, s'\mathrm{F}$.

$$s \,\epsilon\, \mathrm{K} \,.\supset ::\, \varphi \,\epsilon\, s\,'\mathrm{F} \,.=\therefore\, x, y \,\epsilon\, s \,.\, x = y :\supset_{x, y} .\, x\varphi = y\varphi.$$

Exempla. Sit a numerus; tunc $a +$ est functionis praesignum in numerorum classe, et $+ a$ est functionis postsignum; quicumque enim est numerus x, formulae $a + x$ et $x + a$ novos indicant numeros, et ex $x = y$ deducitur $a + x = a + y$, et $x + a = y + a$. Itaque

$$a \,\epsilon\, \mathrm{N} \,.\supset :\, a + .\,\epsilon\,.\, \mathrm{F}' \mathrm{N}.$$
$$a \,\epsilon\, \mathrm{N} \,.\supset :\, + a \,.\,\epsilon\,.\, \mathrm{N}\,'\mathrm{F}.$$

Sit φ functionis praesignum in classe s. Tunc $[\varphi]\, y$ classem significat iis x constitutam, quae conditioni $\varphi\, x = y$ satisfaciunt; scilicet:

Def. $s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, \mathrm{F}' s :\supset : [\varphi]\, y \,.=.\, [x\,\epsilon]\, (\varphi\, x = y).$

Classis $[\varphi]\, y$ vel unum vel plura, vel etiam nullum individuum continere potest. Erit:

$$s \epsilon K . \varphi \epsilon F' s : \supset : y = \varphi x . = . x \epsilon [\varphi] y .$$

Si vero φy uno tantum constat individuo, erit $y = \varphi x . = . x = [\varphi] y$.

Sit φ functionis postsignum; similiter ponimus:

$$s \epsilon K . \varphi \epsilon s 'F : \supset \therefore y [\varphi] = [x \epsilon] (x \varphi = y).$$

Signum [ ] dicitur *inversionis signum*, eiusque usus nonnullos in logica iam exposuimus. Nam si α est propositio indeterminatum continens x, atque a est classis individuis x composita quae conditioni α satisfaciunt, erit $x \epsilon a . = \alpha$, tunc $a = [x \epsilon]\, \alpha$, ut in V, 1.

Sit α formula indeterminatum continens x, sitque φ functionis praesignum, quod litterae x praepositum, formulam α gignat; scilicet sit $\alpha = \varphi x$; tunc erit $\varphi = \alpha [x]$, et si x' est novum ens, erit $\varphi x' = \alpha [x] x'$, scilicet, si α est formula indeterminatum continens x, tunc $\alpha\ [x]\ x'$ significat id quod obtinetur si in α, loco x, x' ponatur.

Similiter, sit α formula indeterminatum continens x, sitque φ functionis postsignum, ut $x \varphi = \alpha$; deducitur $\varphi = [x]\, \alpha$; tunc, si x' est novum ens, erit $x' \varphi = x' [x]\, \alpha$, scilicet $x'\ [x]\ \alpha$ rursum indicat id quod obtinetur si in α, loco x, x' legatur, ut in V, 4.

Alios quoque usus in logica signum [ ] habere potest, quos breviter esponimus, quum ipsis non utamur. Sint a et b duae classes; tunc $[a \frown\ ]b$ sive $b[\frown a]$ classes indicat x, quae conditioni $b = a \frown x$, sive $b = x \frown a$ satisfaciunt. Si b in a non continetur, nulla classis huic conditioni satisfacit; si b in a continetur, signum $b[\frown a]$ omnes indicat classes quae b continent atque in $b \smile - a$ continentur.

In Arithmetica, sint a, b numeri; tunc $[b + a]$ sive $[a +]\, b$ numerum indicat x, qui conditioni $b = x + a$, sive $b = a + x$ satisfacit, nempe $b - a$. Similiter erit $b\,[\times a] = [a \times]\, b = b/a$. Et in analysi hoc signum usuvenire potest; itaque

$$y = \sin x . = . x \epsilon [\sin]\, y \qquad \text{(loco } x = \text{arc} \sin y)$$

$$d F(x) = f(x)\, dx . = . F(x) \epsilon [d]\, f(x)\, dx \qquad \text{(loco } F(x) = \int f(x)\, dx).$$

Sit rursum φ functionis praesignum in classi s, sitque k classis in s contenta; tunc $\varphi\, k$ classem indicat omnibus $\varphi\, x$ compositam, ubi x sunt entia classis k; scilicet

Def. $s \epsilon \mathrm{K} . k \epsilon \mathrm{K} . k \supset s . \varphi \epsilon \mathrm{F}' s : \supset . \varphi k = [y \epsilon] (k . [\varphi] y : -= \Lambda)$.

Sive $s \epsilon \mathrm{K} . k \epsilon \mathrm{K} . k \supset s . \varphi \epsilon \mathrm{F}' s : \supset . \varphi\, k = [y \epsilon] ([x \epsilon] : x \epsilon k . \varphi x = y \therefore -= \Lambda)$.

Def. $s \epsilon \mathrm{K} . k \epsilon \mathrm{K} . k \supset s . \varphi \epsilon s ' \mathrm{F} : \supset . k \varphi = [y \epsilon] (k . y [\varphi] : -= \Lambda)$.

Itaque, si $\varphi \epsilon \mathrm{F}' s$, tunc $\varphi\, s$ classem indicat omnibus $\varphi\, x$ constitutam, ubi x sint entia classis s. Erit:

$s \epsilon \mathrm{K} . \varphi \epsilon \mathrm{F}' s . y \epsilon \varphi s : \supset : \varphi [\varphi] y = y$.

$s \epsilon \mathrm{K} . a, b \epsilon \mathrm{K} . a \supset s . b \supset s . \varphi \epsilon \mathrm{F}' s : \supset . \varphi (a \cup b) = (\varphi a) \cup (\varphi b)$.

$s \epsilon \mathrm{K} . \varphi \epsilon \mathrm{F}' s : \supset . \varphi \Lambda = \Lambda$.

$s \epsilon \mathrm{K} . a, b \epsilon \mathrm{K} . b \supset s . a \supset b . \varphi \epsilon \mathrm{F}' s : \supset . \varphi a \supset \varphi b$.

$s \epsilon \mathrm{K} . a, b \epsilon \mathrm{K} . a \supset s . b \supset s . \varphi \epsilon \mathrm{F}' s : \supset . \varphi (ab) \supset (\varphi a)(\varphi b)$.

Sit a quaedam classis; tunc $a \cap \mathrm{K}$, sive $\mathrm{K} \cap a$, sive $\mathrm{K}\, a$, classes omnes indicat formae $a \cap x$, sive $x \cap a$, $x\, a$, ubi x est classis quaecumque; scilicet $\mathrm{K}\, a$ indicat classes quae in a continentur. Formula $x \epsilon \mathrm{K}\, a$ idem significat quod $x \epsilon \mathrm{K} . x \supset a$. Hac conventione quandoque utimur; ita $\mathrm{K\, N}$ significat *numerorum classem*.

Similiter, si a est classis, $\mathrm{K} \cup a$ indicat classes quae a continent.

Sit a numerus; tunc $a + \mathrm{N}$, sive $\mathrm{N} + a$, *numeros* indicat *numero a maiores;* $a \times \mathrm{N}$, sive $\mathrm{N} \times a$, sive $\mathrm{N} a$ indicat *multiplices numeri a*; a^{N} indicat *potestates numeri a*; $\mathrm{N}^2, \mathrm{N}^3, \ldots$ indicant *numeros quadratos*, vel *numeros cubos*, etc.

Functionum signorum aequalitatem, productum, potestates, ita definire licet:

Def. $s \epsilon \mathrm{K} . \varphi, \psi \epsilon \mathrm{F}' s : \supset \therefore \varphi = \psi : = : x \epsilon s . \supset . \varphi x = \psi x$.

Def. $s \epsilon \mathrm{K} . \varphi \epsilon \mathrm{F}' s . \psi \epsilon \mathrm{F}' \varphi s . x \epsilon s : \supset . \psi \varphi x = \psi (\varphi x)$.

Itaque, in definitionis hypothesi, erit $\psi \varphi$ novum functionis praesignum; idque *productum signorum* ψ *et* φ vocatur.

Similiterque, si φ, ψ sunt functionis postsigna.

Haec valet propositio:

$s \epsilon \mathrm{K} . \varphi \epsilon \mathrm{F}' s . \varphi s \supset s : \supset : \varphi \varphi s \supset s . \varphi \varphi \varphi s \supset s . \mathrm{etc.}$

Functiones $\varphi \varphi$, $\varphi \varphi \varphi, \ldots$ *iteratae* vocantur, et communiter signis $\varphi^2, \varphi^3, \ldots$ indicantur, ut operationis φ potestates.

Si vero φ est functionis postsignum, hac faciliori notatione, absque ambiguitate, uti licet:

Def. $s \epsilon K . \varphi \epsilon s'F . s \varphi \supset s : \supset : \varphi 1 = \varphi . \varphi 2 = \varphi\varphi . \varphi 3 = \varphi\varphi\varphi .$etc.

In definitionis hypothesi, si $m, n \epsilon N$, erit $\varphi (m+n) = (\varphi m) (\varphi n)$; $(\varphi m) n = \varphi (m n)$.

Si hac definitione in Arithmetica utimur, haec invenimus. Numerum qui sequitur numerum a signo faciliori $a+$ indicare possumus; tunc $a+1, a+2,\ldots$ et, si b est numerus, $a+b$, sensum habent $a+, a++,\ldots$ quod a definitione in § 1 patet. Propositionem 6 in § 1 scribere possumus $N+ \supset N$. Si a, b, c sunt numeri, tunc $a :+ b . c$ significat $a + bc$, et $a : \times b . c$ significat $a b^c$.

Multis aliis proprietatibus gaudent functionum signa, praesertim si conditioni satisfaciunt: $\varphi x = \varphi y . \supset . x = y$. Functionis signum quod huic conditioni satisfacit vocatur a clarissimo Dedekind *simile* (ähnlich Abbildung).

Sed his exponendis locus deest.

Declarationes.

Definitio, vel breviter *Def.* est propositio formam habens $x = a$, sive $\alpha \supset . x = a$, ubi a est signorum aggregatus sensum habens notum; x est signum, vel signorum aggregatus significatione adhuc carens; α vero est conditio sub qua definitio datur.

Theorema, (Theor. vel Th) est propositio quae demonstratur. Si theorema formam habet $\alpha \supset \beta$, ubi α et β sunt propositiones, tunc α dicitur *Hypothesis* (Hyp. vel breviter Hp.), β vero *Thesis* (Thes. vel Ts.). Hyp. ac Ts. a Theorematis forma pendent; nam si loco $\alpha \supset \beta$ scribemus $-\beta \supset -\alpha$, erit $-\beta$ Hp, et $-\alpha$ Ts.; si vero scribemus $\alpha - \beta = \Lambda$, Hp. ac Ts. absunt.

In quolibet § signum P quod quidam numerus sequatur, propositionem indicat eiusdem § hoc numero signatam. Logicae propositiones indicantur signo L et propositionis numero.

Formulae quae in una linea non continentur, in altera linea, nullo interposito signo, sequuntur.

ARITHMETICES PRINCIPIA.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur *numerus (integer positivus).*
» 1 » *unitas.*
» $a+1$ » *sequens a,* sive *a plus* 1.
» $=$ » *est aequalis.* Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.

Axiomata.

1. $1 \in N.$
2. $a \in N . \supset . a = a.$
3. $a, b, c \in N . \supset : a = b . = . b = a.$
4. $a, b \in N . \supset \therefore a = b . b = c : \supset . a = c.$
5. $a = b . b \in N : \supset . a \in N.$
6. $a \in N . \supset . a + 1 \in N.$
7. $a, b \in N . \supset : a = b . = . a + 1 = b + 1.$
8. $a \in N . \supset . a + 1 -= 1.$
9. $k \in K \therefore 1 \in k \therefore x \in N . x \in k : \supset_x . x + 1 \in k :: \supset . N \supset k.$

Definitiones.

10. $2 = 1 + 1;\ 3 = 2 + 1;\ 4 = 3 + 1;$ etc.

Theoremata.

11. $2 \epsilon N$.

Demonstratio:

$P1 . \supset :$	$1 \epsilon N$	(1)
$1 [a] (P6) . \supset :$	$1 \epsilon N . \supset . 1 + 1 \epsilon N$	(2)
$(1)(2) . \supset :$	$1 + 1 \epsilon N$	(3)
$P10 . \supset :$	$2 = 1 + 1$	(4)
$(4) . (3) . (2, 1 + 1) [a, b] (P5) : \supset :$	$2 \epsilon N$	(Theorema).

Nota. — Huius facillimae demonstrationis gradus omnes explicite scripsimus. Brevitatis causa ipsam ita scribemus:

$$P1 . 1 [a] (P6) : \supset : 1 + 1 \epsilon N . P10 . (2, 1+1) [a, b] (P5) : \supset : \mathrm{Th.}$$

vel

$$P1 . P6 : \supset : 1 + 1 \epsilon N . P10 . P5 : \supset : \mathrm{Th.}$$

12. $3, 4, \ldots \epsilon N$.

13. $a, b, c, d \epsilon N . a = b . b = c . c = d : \supset : a = d$.

Dem. Hyp. $P4 : \supset : a, c, d \epsilon N . a = c . c = d . P4 : \supset :$ Thes.

14. $a, b, c \epsilon N . a = b . b = c . a -= c : = \Lambda$.

Dem. $P4 . L39 : \supset .$ Theor.

15. $a, b, c \epsilon N . a = b . b -= c : \supset . a -= c$.

16. $a, b \epsilon N . a = b : \supset . a + 1 = b + 1$.

16′. $a, b \epsilon N . a + 1 = b + 1 : \supset . a = b$.

Dem. $P7 = (P16)(P16')$.

17. $a, b \epsilon N . \supset : a -= b . = . a + 1 -= b + 1$.

Dem. $P7 . L21 : \supset .$ Theor.

Definitio.

18. $a, b \epsilon N . \supset . a + (b + 1) = (a + b) + 1$.

Nota. — Hanc definitionem ita legere oportet: si a et b sunt numeri, et $(a + b) + 1$ sensum habet (scilicet si $a + b$ est numerus), sed $a + (b + 1)$ nondum definitus est, tunc $a + (b + 1)$ significat numerum qui $a + b$ sequitur.

Ab hac definitione, et a praecedentibus deducitur:

$$a \epsilon N . \supset \therefore a + 2 = a + (1 + 1) = (a + 1) + 1.$$

$$a \epsilon N . \supset \therefore a + 3 = a + (2 + 1) = (a + 2) + 1, \text{etc.}$$

Theoremata.

19. $a, b \in N . \supset . a + b \in N.$

Dem. $a \in N . P6 : \supset : a + 1 \in N : \supset : 1 \in [b \in] Ts.$ (1)

$a \in N . \supset :: b \in N . b \in [b \in] Ts : \supset : a + b \in N . P6 : \supset : (a + b) + 1 \in N . P18 : \supset : a + (b + 1) \in N : \supset : (b + 1) \in [b \in] Ts.$ (2)

$a \in N . (1) . (2) . \supset :: 1 \in [b \in] Ts \therefore b \in N . b \in [b \in] Ts : \supset : b + 1 \in [b \in] Ts \therefore ([b \in] Ts) [k] P9 :: \supset : N \supset [b \in] Ts . (L50) :: \supset : b \in N . \supset Ts.$ (3)

$(3) . (L42) : \supset : a, b \in N . \supset .$ Thesis. (Theor.).

20. *Def.* $a + b + c = (a + b) + c.$

21. $a, b, c \in N . \supset . a + b + c \in N.$

22. $a, b, c \in N . \supset : a = b . = . a + c = b + c.$

Dem. $a, b \in N . P7 : \supset . 1 \in [c \in] Ts.$ (1)

$a, b \in N . \supset :: c \in N . c \in [c \in] Ts \therefore \supset \therefore a = b . = . a + c = b + c : a + c, b + c \in N : a + c = b + c . = . a + c + 1 = b + c + 1 \therefore \supset \therefore a = b . = . a + (c + 1) = b + (c + 1) \therefore \supset \therefore (c + 1) \in [c \in] Ts.$ (2)

$a, b \in N . (1) . (2) : \supset :: 1 \in [c \in] Ts \therefore c \in [c \in] Ts . \supset . (c + 1) \in [c \in] Ts :: \supset :: c \in N . \supset . Ts.$ (3)

$(3) \supset$ Theor.

23. $a, b, c \in N . \supset . a + (b + c) = a + b + c.$

Dem. $a, b \in N . P18 . P20 : \supset . 1 \in [c \in] Ts.$ (1)

$a, b \in N . \supset \therefore c \in N . c \in [c \in] Ts : \supset : a + (b + c) = a + b + c . P7 : \supset : a + (b + c) + 1 = a + b + c + 1 . P18 : \supset : a + (b + (c + 1)) = a + b + (c + 1) : \supset : c + 1 \in [c \in] Ts.$ (2)

$(1) (2) (P9) . \supset .$ Theor.

24. $a \in N . \supset . 1 + a = a + 1.$

Dem. $P2 . \supset . 1 \in [a \in] Ts.$ (1)

$a \in N . a \in [a \in] Ts : \supset : 1 + a = a + 1 : \supset : 1 + (a + 1) = (a + 1) + 1 : \supset : (a + 1) \in [a \in] Ts.$ (2)

$(1) (2) . \supset .$ Theor.

24′. $a, b \in N . \supset . 1 + a + b = a + 1 + b.$

Dem. Hyp. $P24 : \supset : 1 + a = a + 1 . P22 : \supset .$ Thesis.

25. $a, b \epsilon \mathrm{N} . \supset . a + b = b + a.$

Dem. $a \epsilon \mathrm{N} . \mathrm{P}\,24 : \supset : 1 \epsilon [b \epsilon]\, \mathrm{Ts}.$ (1)

$a \epsilon \mathrm{N} . \supset \therefore b \epsilon \mathrm{N} . b \epsilon [b \epsilon]\, \mathrm{Ts} : \supset : a + b = b + a . \mathrm{P}\,7 : \supset : (a + b) + 1 = (b + a) + 1 . (a + b) + 1 = a + (b + 1) . (b + a) + 1 = 1 + (b + a) . 1 + (b + a) = (1 + b) + a . (1 + b) + a = (b + 1) + a : \supset : a + (b + 1) = (b + 1) + a : \supset : (b + 1) \epsilon [b \epsilon]\, \mathrm{Ts}.$ (2)

$(1)(2) . \supset .$ Theor.

26. $a, b, c \epsilon \mathrm{N} . \supset : a = b . = . c + a = c + b.$

27. $a, b, c \epsilon \mathrm{N} . \supset : a + b + c = a + c + b.$

28. $a, b, c, d \epsilon \mathrm{N} . a = b . c = d : \supset . a + c = b + d.$

§ 2. De subtractione.

Explicationes.

Signum $-$ legitur *minus*.

» $<$ » *est minor*.

» $>$ » *est maior*.

Definitiones.

1. $a, b \epsilon \mathrm{N} . \supset : b - a = \mathrm{N}\, [x \epsilon]\, (x + a = b).$

2. $a, b \epsilon \mathrm{N} . \supset : a < b . = . b - a -= \Lambda.$

3. $a, b \epsilon \mathrm{N} . \supset : b > a . = . a < b.$

$a + b - c = (a + b) - c ; a - b + c = (a - b) + c ; a - b - c = (a - b) - c.$

Theoremata.

4. $a, b, a', b' \epsilon \mathrm{N} . a = a' . b = b' : \supset : b - a = b' - a'.$

Dem. $\mathrm{Hyp} . \supset : x + a = b . = . x + a' = b' : \supset .$ Thesis.

5. $a, b \epsilon \mathrm{N} . \supset : a < b . = . b - a \epsilon \mathrm{N}.$

Dem. $a, b \epsilon \mathrm{N} : \supset \therefore x, y \epsilon b - a . \supset_{x, y} : x, y \epsilon \mathrm{N} . x + a = b . y + a = b . \S 1\, \mathrm{P}\,22 : \supset : x = y.$ (1)

$a, b \epsilon \mathrm{N} . a < b . \mathrm{P}\,2 . (1) : \supset \therefore b - a -= \Lambda : x, y \epsilon b - a . \supset . x = y : (\mathrm{N}, b - a)\, [s, k]\, (\mathrm{L}\,56) \therefore \supset \therefore b - a \epsilon \mathrm{N}.$ (2)

$a, b \in N . b - a \in N . (L 56) : \supset : b - a - = \Lambda : \supset : a < b.$ (3)

(2)(3) . $\supset$. Theor.

6. $a, b \in N . a < b : \supset . b - a + a = b.$

Dem. Hyp . P 5 . P 1 : $\supset : b - a \in N . (b - a) \in [x \in] (x + a = b) : \supset :$ Thes.

7. $a, b, c \in N . \supset : c = b - a . = . c + a = b.$

Dem. Hyp . § 1 P 22 . P 6 : $\supset : c = b - a . = . c + a = b - a + a . = . c + a = b.$

8. $a, b \in N . \supset . a + b - a = b.$

Dem. $(a + b, b) [b, c]$ P 7 . $\supset$. Theor.

9. $a, b, c \in N . a < b : \supset : c + (b - a) = c + b - a.$

Dem. Hyp . P 6 : $\supset : (b - a) + a = b : \supset : c + (b - a) + a = c + b .$ P 7 : $\supset$: Thesis.

10. $a, b, c \in N . a > b + c : \supset . a - (b + c) = a - b - c.$

11. $a, b, c \in N . b > c . a > b - c : \supset . a - (b - c) = a + c - b.$

12. $a, b, a', b' \in N . a = a' . b = b' : \supset : a < b . = . a' < b'.$

Dem. Hyp . $\supset . b - a = b' - a' . \supset . b - a \in N = b' - a' \in N . \supset$. Thes.

13. $a, b \in N . \supset . a < a + b.$

Dem. Hyp . P 8 : $\supset : a + b - a = b : \supset . a + b - a \in N .$ P 5 : $\supset$: Thesis.

14. $a, b, c \in N . a < b . b < c : \supset . a < c.$

Dem. Hyp . $\supset : b - a \in N . c - b \in N : \supset : (b - a) + (c - b) \in N : \supset : c - a \in N : \supset$. Thesis.

15. $a, b, c \in N . \supset : a < b . = . a + c < b + c.$

Dem. Hyp . $\supset : a < b . = . b - a \in N . = . (b + c) - (a + c) \in N . = . a + c < b + c.$

16. $a, b, a', b' \in N . a < b . a' < b' : \supset . a + a' < b + b'.$

Dem. Hyp . $\supset : a + a' < b + a' . b + a' < b + b' : \supset$. Thesis.

17. $a, b, c \in N . a < b < c : \supset . c - a > c - b.$

Dem. Hyp . $\supset . b - a \in N . c - b \in N . (c - b) + (b - a) = c - a : \supset$. Thesis.

18. $a \in N . \supset : a = 1 . \cup . a > 1.$

Dem. $1 \in [a \in]$ Thesis. (1)

$a \in N .$ P 13 : $\supset : a + 1 > 1 : \supset : a + 1 \in [a \in]$ Thesis. (2)

(1)(2) . $\supset$. Theor.

19. $a, b \varepsilon N . \supset . a + b -= b.$

Dem. $a \varepsilon N . \S 1 \, P 8 : \supset : a + 1 -= 1 : \supset : 1 \varepsilon [b \varepsilon]$ Thesis. (1)

$a \varepsilon N . b \varepsilon N . b \varepsilon [b \varepsilon] \, Ts : \supset : a + b -= b . \S 1 \, P 17 : \supset : a + (b + 1) -= b + 1 : \supset : b + 1 \varepsilon [b \varepsilon]$ Ts. (2)

(1)(2) . $\supset$. Theor.

20. $a, b \varepsilon N . a < b . a = b : = \Lambda.$

Dem. $\text{Hyp} : \supset : b - a \varepsilon N . (b - a) + a = a . P 19 : \supset : \Lambda.$

21. $a, b \varepsilon N . a > b . a = b : = \Lambda.$

22. $a, b \varepsilon N . a > b . a < b : = \Lambda.$

23. $a, b \varepsilon N : \supset : a < b . \cup . a = b . \cup . a > b.$

Dem. $a \varepsilon N . P 18 : \supset . 1 \varepsilon [b \varepsilon]$ Ts. (1)

$a, b \varepsilon N . a < b : \supset . a < b + 1.$ (2)

$a, b \varepsilon N . a = b : \supset . a < b + 1.$ (3)

$a, b \varepsilon N . a > b : \supset : a - b \varepsilon N . P 18 : \supset : a - b = 1 . \cup . a - b > 1.$ (4)

$a, b \varepsilon N . a - b = 1 : \supset . a = b + 1.$ (5)

$a, b \varepsilon N . a - b > 1 : \supset . a > b + 1.$ (6)

$a, b \varepsilon N . a > b . (4)(5)(6) : \supset : a = b + 1 . \cup . a > b + 1.$ (7)

$a, b \varepsilon N : a < b . \cup . a = b . \cup . a > b : (2)(3)(7) \therefore \supset \therefore a < b + 1 . \cup . a = b + 1 . \cup . a > b + 1.$ (8)

$a, b \varepsilon N . b \varepsilon [b \varepsilon] \, Ts . (8) : \supset : b + 1 \varepsilon [b \varepsilon]$ Ts. (9)

(1)(9) . $\supset$. Theor.

§ 3. De maximis et minimis.

Explicationes.

Sit $a \varepsilon K N$, hoc est sit a quaedam numerorum classis; tunc Ma legatur *maximus inter* a, et Ɯa legatur *minimus inter* a.

Definitiones.

1. $a \varepsilon K N . \supset : M a = [x \varepsilon] (x \varepsilon a \therefore a . \ni > x : = \Lambda).$

2. $a \varepsilon K N . \supset . \text{Ɯ} a = [x \varepsilon] (x \varepsilon a \therefore a . \ni < x : = \Lambda).$

Theoremata.

3. $n \epsilon \mathrm{N} . a \epsilon \mathrm{K}\,\mathrm{N} . a -= \Lambda . a \ni > n = \Lambda :\supset . \mathrm{M}\, a \epsilon \mathrm{N}.$

Dem. $a \epsilon \mathrm{K}\,\mathrm{N} . a -= \Lambda . a \ni > 1 = \Lambda :\supset: a = 1 :\supset . \mathrm{M}\, a = 1 :\supset . \mathrm{M}\, a \epsilon \mathrm{N}.$ (1)

$(1) \supset : 1 \epsilon [n \epsilon] (\mathrm{Hp} \supset \mathrm{Ts}).$ (2)

$n \epsilon \mathrm{N} . a \epsilon \mathrm{K}\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 \epsilon a :\supset: n + 1 = \mathrm{M}\, a :\supset: \mathrm{M}\, a \epsilon \mathrm{N}.$ (3)

$n \epsilon \mathrm{N} . a \epsilon \mathrm{K}\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 -\epsilon\, a :\supset: a \ni > n = \Lambda.$ (4)

$n \epsilon [n \epsilon] (\mathrm{Hp} \supset \mathrm{Ts}) . a \epsilon \mathrm{K}\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 -\epsilon\, a :\supset: \mathrm{M}\, a \epsilon \mathrm{N}.$ (5)

$n \epsilon [n \epsilon] (\mathrm{Hp} \supset \mathrm{Ts}) . a \epsilon \mathrm{K}\,\mathrm{N} . a \ni > n + 1 = \Lambda . (3)(5) :\supset: \mathrm{M}\, a \epsilon \mathrm{N}.$ (6)

$n \epsilon [n \epsilon] (\mathrm{Hp} \supset \mathrm{Ts}) . (6) :\supset . (n + 1) \epsilon [n \epsilon] (\mathrm{Hp} \supset \mathrm{Ts}).$ (7)

$(2)(7) . \S 1\, \mathrm{P}\, 9 :\supset: n \epsilon \mathrm{N} . \supset . \mathrm{Hp} \supset \mathrm{Ts}.$ (Theor.)

4. $a \epsilon \mathrm{K}\,\mathrm{N} . a -= \Lambda :\supset . \mathrm{W}\, a \epsilon \mathrm{N}.$

5. $a \epsilon \mathrm{K}\,\mathrm{N} . \supset : \mathrm{W}\, a = \mathrm{M}\, [x \epsilon] (a \ni < x = \Lambda).$

§ 4. De multiplicatione.

Definitiones.

1. $a \epsilon \mathrm{N} . \supset . a \times 1 = a.$

2. $a, b \epsilon \mathrm{N} . \supset . a \times (b + 1) = a \times b + a.$

$ab = a \times b \,;\, ab + c = (ab) + c \,;\, abc = (ab)\, c.$

Theoremata.

3. $a, b \epsilon \mathrm{N} . \supset . ab \epsilon \mathrm{N}.$

Dem. $a \epsilon \mathrm{N} . \mathrm{P}\, 1 :\supset: a \times 1 \epsilon \mathrm{N} :\supset . 1 \epsilon [b \epsilon] \mathrm{Ts}.$ (1)

$a, b \epsilon \mathrm{N} . b \epsilon [b \epsilon] \mathrm{Ts} :\supset: a \times b \epsilon \mathrm{N} . \S 1\, \mathrm{P}\, 19 :\supset: ab + a \epsilon \mathrm{N} . \mathrm{P}\, 1 :\supset: a (b + 1) \epsilon \mathrm{N} :\supset: b + 1 \epsilon [b \epsilon] \mathrm{Ts}.$ (2)

$(1)(2) . \supset . \mathrm{Theor.}$

4. $a, b, c \in \mathrm{N} . \supset . (a + b) c = ac + bc.$

Nota. Haec est prop. 5ª EUCLIDIS elem. libri VII.

Dem. $a, b \in \mathrm{N} . \mathrm{P}\,1 : \supset : 1 \in [c \in] \mathrm{Ts}.$ (1)

$a, b, c \in \mathrm{N} . c \in [c \in] \mathrm{Ts} : \supset : (a + b) c = ac + bc . \S 1\ \mathrm{P}\,22 : \supset : (a + b) c + a + b = ac + bc + a + b . \mathrm{P}\,2 : \supset : (a + b) (c + 1) = a (c + 1) + b (c + 1) : \supset : c + 1 \in [c \in] \mathrm{Ts}.$ (2)

(1) (2) . $\supset$. Theor.

5. $a \in \mathrm{N} . \supset . 1 \times a = a.$

Dem. $1 \in [a \in] \mathrm{Ts}.$ (1)

$a \in [a \in] \mathrm{Ts} . \supset . 1 \times a = a . \supset . 1 \times a + 1 = a + 1 . \supset . 1 \times (a + 1) = a + 1 . \supset . a + 1 \in [a \in] \mathrm{Ts}.$ (2)

(1) (2) . $\supset$. Theor.

6. $a, b \in \mathrm{N} . \supset . b a + a = (b + 1) a.$

7. $a, b \in \mathrm{N} . \supset . ab = ba.$ (EUCL. VII, 16)

Dem. $a \in \mathrm{N} . \mathrm{P}\,5 . \mathrm{P}\,1 : \supset . a \times 1 = a = 1 \times a : \supset : 1 \in [b \in] \mathrm{Ts}.$ (1)

$a, b \in \mathrm{N} . b \in [b \in] \mathrm{Ts} : \supset : ab = ba : \supset : ab + a = ba + a . \mathrm{P}\,1 . \mathrm{P}\,6 : \supset : a (b + 1) = (b + 1) a : \supset : b + 1 \in [b \in] \mathrm{Ts}.$ (2)

(1) (2) . $\supset$. Theor.

8. $a, b, c \in \mathrm{N} . \supset . a (b + c) = ab + ac.$

Dem. P 4 . P 7 : $\supset$. Theor.

9. $a, b, c \in \mathrm{N} . a = b : \supset : ac = bc.$

Dem. $a, b \in \mathrm{N} . a = b :: \supset :: 1 \in [c \in] \mathrm{Ts} \therefore c \in [c \in] \mathrm{Ts} . \supset : ac = bc . a = b : \supset : ac + a = bc + b : \supset : a (c + 1) = b (c + 1) : \supset : c + 1 \in [c \in] \mathrm{Ts} :: \supset : c \in \mathrm{N} . \supset . \mathrm{Ts}.$

10. $a, b, c \in \mathrm{N} : a < b : \supset . (b - a) c = bc - ac.$ (EUCL. VII, 7)

Dem. $\mathrm{Hyp} . \supset : b - a \in \mathrm{N} . (b - a) + a = b : \supset : (b - a) c + ac = bc : \supset : (b - a) c = bc - ac.$

11. $a, b, c \in \mathrm{N} . a < b : \supset : ac < bc.$

Dem. $\mathrm{Hyp} . \supset : b - a \in \mathrm{N} . \mathrm{P}\,3 : \supset : (b - a) c \in \mathrm{N} . \mathrm{P}\,10 : \supset : bc - ac \in \mathrm{N} : \supset$ Thesis.

12. $a, b, c \in \mathrm{N} . \supset \therefore a < b . = . ac < bc : a = b . = . ac = bc : a > b . = . ac > bc.$

13. $a, b, a', b' \in \mathrm{N} . a < a' . b < b' : \supset : ab < a' b'.$

14. $a, b \in \mathrm{N} : \supset . ab . > \cup = . a.$

15. $a, b, c \in \mathrm{N} . \supset . a (bc) = abc.$

Dem. $a, b \in N . P1 : \supset : 1 \in [c \in] Ts.$ (1)

$a, b, c \in N . c \in [c \in] Ts : \supset : a(bc) = abc : \supset : a(bc) + ab = abc + ab : \supset : a(bc + b) = ab(c+1) : \supset : a(b(c+1)) = ab(c+1) : \supset : c + 1 \in [c \in] Ts.$ (2)

$(1)(2) . \supset .$ Theor.

§ 5. De potestatibus.

Definitiones.

1. $a \in N . \supset . a^1 = a.$
2. $a, b \in N . \supset . a^{b+1} = a^b a.$

Theoremata.

3. $a, b \in N . \supset . a^b \in N.$

Dem. $a \in N . P1 : \supset . 1 \in [b \in] Ts.$ (1)

$a, b \in N . b \in [b \in] Ts : \supset : a^b \in N . §4 P3 : \supset : a^b a \in N . P1 : \supset : a^{b+1} \in N : \supset : b + 1 \in [b \in] Ts.$ (2)

$(1)(2) . \supset .$ Theor.

4. $a \in N . \supset . 1^a = 1.$
5. $a, b, c \in N . \supset . a^{b+c} = a^b a^c.$
6. $a, b, c \in N . \supset . (ab)^c = a^c b^c.$
7. $a, b, c \in N . \supset . (a^b)^c = a^{bc}.$
8. $a, b, c \in N . \supset \therefore a < b . = . a^c < b^c : a = b . = . a^c = b^c : a > b . = . a^c > b^c.$
9. $a, b, c \in N . a > 1 . \supset \therefore b < c . = . a^b < a^c : b = c . = . a^b = a^c : b > c . = . a^b > a^c.$

§ 6. De divisione.

Explicationes.

Signum / legatur *divisus per*.

» D » *dividit*, sive *est divisor*.

» ꓷ » *est multiplex*.

» Np » *numerus primus*.

» π » *est primus cum*.

Definitiones.

1. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset : b \,/\, a = \mathrm{N}\,[x \,\epsilon]\,(xa = b).$
2. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset : a \,\mathrm{D}\, b \,.=.\, b \,/\, a - = \Lambda.$
3. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset : b \,\text{ɑ}\, a \,.=.\, a \,\mathrm{D}\, b.$
4. $\mathrm{Np} = \mathrm{N}\,[x \,\epsilon]\,(\ni \mathrm{D}\, x \,.\, \ni > 1 \,.\, \ni < x : = \Lambda).$
5. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset :: a \,\pi\, b \therefore = \therefore \ni \mathrm{D}\, a \,.\, \ni \mathrm{D}\, b \,.\, \ni > 1 : = \Lambda.$
6. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset \therefore \ni \mathrm{D}\,(a, b) : = : \ni \mathrm{D}\, a \,.\cap.\, \ni \mathrm{D}\, b.$
7. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset \therefore \ni \text{ɑ}\,(a, b) : = : \ni \text{ɑ}\, a \,.\cap.\, \ni \text{ɑ}\, b.$

$ab/c = (ab)/c;\; a/b/c = (a/b)/c;\; a/b \times c = (a/b)\,c.$

Theoremata.

Nota. Haec theoremata ut in subtractione demonstrantur.

8. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \supset .\, a \,/\, b = a' \,/\, b'.$
9. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \supset : a \,\mathrm{D}\, b \,.=.\, a' \,\mathrm{D}\, b'.$
10. $a, b, c \,\epsilon\, \mathrm{N} \,.\, \supset : ac = b \,.=.\, c = b \,/\, a.$
11. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset : a \,\mathrm{D}\, b \,.=.\, b \,/\, a \,\epsilon\, \mathrm{N}.$
12. $a \,\epsilon\, \mathrm{N} \,.\, \supset .\, a \,/\, 1 = a.$
13. $a \,\epsilon\, \mathrm{N} \,.\, \supset .\, a \,/\, a = 1.$
14. $a \,\epsilon\, \mathrm{N} \,.\, \supset .\, 1 \,\mathrm{D}\, a.$
15. $a \,\epsilon\, \mathrm{N} \,.\, \supset .\, a \,\mathrm{D}\, a.$
16. $a, b \,\epsilon\, \mathrm{N} \,.\, ab \,/\, b = a.$
17. $a, b \,\epsilon\, \mathrm{N} \,.\, a \,\mathrm{D}\, b : \supset .\, a\,(b \,/\, a) = b.$
18. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, b : \supset .\, a\,(b \,/\, c) = ab \,/\, c.$
19. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\text{ɑ}\, bc : \supset : a \,/\, (bc) = a \,/\, b \,/\, c.$
20. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\text{ɑ}\, b \,.\, b \,\text{ɑ}\, c : \supset .\, a \,/\, (b \,/\, c) = a \,/\, b \times c.$
21. $a, m, n \,\epsilon\, \mathrm{N} \,.\, m > n : \supset .\, a^m / a^n = a^{m-n}.$
22. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset .\, a \,\mathrm{D}\, ab.$
23. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\mathrm{D}\, b \,.\, b \,\mathrm{D}\, c : \supset .\, a \,\mathrm{D}\, c.$
24. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\mathrm{D}\, b \,\mathrm{D}\, c : \supset .\, c \,/\, a \,\text{ɑ}\, c \,/\, b.$
25. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, b : \supset .\, (a + b) \,/\, c = a \,/\, c + b \,/\, c.$
26. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, b \,.\, a > b : \supset : (a - b) \,/\, c = a \,/\, c - b \,/\, c.$
27. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, b : \supset .\, c \,\mathrm{D}\, a + b.$
28. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, b \,.\, a > b : \supset .\, c \,\mathrm{D}\, a - b.$

29. $a, b, c, m, n \,\epsilon\, \mathrm{N} . c \,\mathrm{D}\, a . c \,\mathrm{D}\, b : \supset . c \,\mathrm{D}\, ma + nb.$

30. $a, b, c, m, n \,\epsilon\, \mathrm{N} . c \,\mathrm{D}\, a . c \,\mathrm{D}\, b . m a > n b : \supset . c \,\mathrm{D}\, ma - nb.$

31. $a, b \,\epsilon\, \mathrm{N} . a \,\mathrm{D}\, b : \supset : a . < \cup = . b.$

Dem. Hyp. P 11 . P 17 . § 4 P 14 $: \supset : b / a \,\epsilon\, \mathrm{N} . a (b / a) = b . a < \cup =$
$a (b / a) : \supset .$ Thesis.

32. $a, b \,\epsilon\, \mathrm{N} . a \,\mathrm{D}\, b . b \,\mathrm{D}\, a : \supset . a = b.$

33. $a \,\epsilon\, \mathrm{N} . \supset . \mathrm{M} \ni \mathrm{D}\, a = a.$

34. $a, b \,\epsilon\, \mathrm{N} . a > b : \supset . \ni \mathrm{D}\, (a, b) = \ni \mathrm{D}\, (b, a - b).$

Dem. Hyp. P 28 $: \supset \therefore x \,\mathrm{D}\, a . x \,\mathrm{D}\, b : \supset : x \,\mathrm{D}\, b . x \,\mathrm{D}\, (a - b)$ (1)
Hyp. P 27 $: \supset \therefore x \,\mathrm{D}\, b . x \,\mathrm{D}\, (a - b) : \supset : x \,\mathrm{D}\, b . x \,\mathrm{D}\, (b + (a - b))$
$: \supset : x \,\mathrm{D}\, b . x \,\mathrm{D}\, a.$ (2)
(1) (2) $\supset :$ Hyp. $\supset \therefore x \,\mathrm{D}\, a . x \,\mathrm{D}\, b : = : x \mathrm{D} b . x \mathrm{D} (a - b).$ (Theor.)

35. $a, b \,\epsilon\, \mathrm{N} . \supset : \mathrm{M} \ni \mathrm{D}\, (a, b) \,\epsilon\, \mathrm{N}.$

Dem. $1 \,\mathrm{D}\, a . 1 \,\mathrm{D}\, b : \supset : \ni \mathrm{D}\, (a, b) - = \Lambda.$ (1)
$\ni \mathrm{D}\, (a, b) . \ni > a : = \Lambda.$ (2)
(1)(2) . § 3 P 3 $: \supset .$ Th.

36. $a, b \,\epsilon\, \mathrm{N} . \supset . \ni \mathrm{D}\, (a, b) = \ni \mathrm{D}\, \mathrm{M} \ni \mathrm{D}\, (a, b).$ (Eucl. VII, 2)

Dem. $k = \mathrm{N}\, [c\, \epsilon]$ (Hp. $a < c . b < c : \supset .$ Ts.). (1)
$a \,\epsilon\, \mathrm{N} . b \,\epsilon\, \mathrm{N} . a < 1 . b < 1 : = \Lambda.$ (2)
(1) (2) $. \supset . 1 \,\epsilon\, k.$ (3)
$a, b \,\epsilon\, \mathrm{N} . a < c + 1 . b < c + 1 : \supset \therefore a < c . b < c : \cup : a = c .$
$b < c : \cup : a < c . b = c : \cup : a = c . b = c.$ (4)
$c \,\epsilon\, k . a, b \,\epsilon\, \mathrm{N} . a < c . b < c : \supset :$ Ts. (5)
$c \,\epsilon\, k . a, b \,\epsilon\, \mathrm{N} . a = c . b < c : \supset : c \,\epsilon\, k . b < c . a - b < c . \ni \mathrm{D}$
$(a, b) = \ni \mathrm{D}\, (b, a - b) : \supset : \ni \mathrm{D}\, (b, a - b) = \ni \mathrm{D} \mathrm{M} \ni \mathrm{D} (b, a - b)$
$: \supset : \ni \mathrm{D}\, (a, b) = \ni \mathrm{D}\, \mathrm{M} \ni \mathrm{D}\, (a, b) : \supset :$ Ts. (6)
$(a, b)\, [b, a]$ (6) $\supset . c \,\epsilon\, k . a, b \,\epsilon\, \mathrm{N} . a < c . b = c : \supset :$ Ts. (7)
$c \,\epsilon\, k . a, b \,\epsilon\, \mathrm{N} . a = c . b = c : \supset : \ni \mathrm{D}\, (a, b) = \ni \mathrm{D} c = \ni \mathrm{D} \mathrm{M} \ni \mathrm{D} c$
$= \ni \mathrm{D}\, \mathrm{M} \ni \mathrm{D}\, (a, b) : \supset :$ Ts. (8)
(4) (5) (6) (7) (8) $. \supset . c \,\epsilon\, k . a, b \,\epsilon\, \mathrm{N} . a < c + 1 . b < c + 1 : \supset :$ Ts. (9)
(9) $\supset . c \,\epsilon\, k . \supset . (c + 1) \,\epsilon\, k.$ (10)
(1) (10) $. \supset \therefore c \,\epsilon\, \mathrm{N} .$ Hp. $a < c . b < c : \supset :$ Ts. (11)
$(a + b)\, [c]$ (11) $. \supset :$ Hp. $\supset .$ Ts. (Theor.)

37. $a, b, m \,\epsilon\, \mathrm{N} . \supset . \mathrm{M} \ni \mathrm{D}\, (am, bm) = m \times \mathrm{M} \ni \mathrm{D}\, (a, b).$

§ 7. Theoremata varia.

1. $a, b \in \mathrm{N} . a^2 + b^2 \,\text{ɑ}\, 7 : \supset : a \,\text{ɑ}\, 7 . b \,\text{ɑ}\, 7.$
2. $x \in \mathrm{N} . \supset . x(x+1) \,\text{ɑ}\, 2.$
3. $x \in \mathrm{N} . \supset . x(x+1)(x+2) \,\text{ɑ}\, 6.$
4. $x \in \mathrm{N} . \supset . x(x+1)(2x+1) \,\text{ɑ}\, 6.$
5. $x \in \mathrm{N} . \supset : x . \pi . x+1.$
6. $x \in \mathrm{N} . \supset : 2x - 1 . \pi . 2x+1.$
7. $x \in \mathrm{N} . \supset . (2x+1)^2 - 1 \,\text{ɑ}\, 8.$
8. $a \in \mathrm{N} . a > 1 : \supset \therefore \mathrm{Np} . \ni > 1 . \ni \mathrm{D}\, a : - = \Lambda.$ (Eucl. VII, 31)
9. $a, b \in \mathrm{N} \therefore b^2 > a \therefore \ni \mathrm{D}\, a . \ni > 1 . \ni < b : = \Lambda :: \supset . a \in \mathrm{Np}.$
10. $a, b \in \mathrm{N} . a \in \mathrm{Np} . a - \mathrm{D}\, b : \supset : a \,\pi\, b.$ (Eucl. VII, 29)
11. $a, b, c \in \mathrm{N} . a \,\mathrm{D}\, bc . a \,\pi\, b : \supset . a \,\mathrm{D}\, c.$
12. $a, b \in \mathrm{N} . m = \mathrm{M} \ni \mathrm{D}(a, b) : \supset : a / m . \pi . b / m.$
13. $a \in \mathrm{Np} . b, c \in \mathrm{N} . a \,\mathrm{D}\, bc : \supset : a \,\mathrm{D}\, b . \cup . a \,\mathrm{D}\, c.$ (Eucl. VII, 30)
14. $a \in \mathrm{Np} . b, n \in \mathrm{N} : \supset : a \,\mathrm{D}\, b^n . = . a \,\mathrm{D}\, b.$ (Eucl. IX, 12)
15. $a, b, c \in \mathrm{N} . a \,\pi\, b . c \,\mathrm{D}\, a : \supset : c \,\pi\, b.$ (Eucl. VII, 23)
16. $a, b, c \in \mathrm{N} . \supset \therefore a \,\pi\, b . a \,\pi\, c : = : a \,\pi\, bc.$ (Eucl. VII, 24)
17. $a, b, c \in \mathrm{N} . b \,\pi\, c . b \,\mathrm{D}\, a . c \,\mathrm{D}\, a : \supset . bc \,\mathrm{D}\, a.$
18. $a, b, c \in \mathrm{N} . a \,\pi\, b : \supset : \ni \mathrm{D}\,(ac, b) = \ni \mathrm{D}\,(c, b).$
19. $a, b \in \mathrm{N} . \supset . \text{W} \ni \text{ɑ}\,(a, b) \in \mathrm{N}.$
20. $a, b \in \mathrm{N} . \supset . \text{W} \ni \text{ɑ}\,(a, b) = ab / \mathrm{M} \ni \mathrm{D}\,(a, b).$ (Eucl. VII, 34)
21. $a, b, c \in \mathrm{N} . c \,\text{ɑ}\, a . c \,\text{ɑ}\, b : \supset : c \,\text{ɑ}\, \text{W} \ni \mathrm{D}\,(a, b).$ (Eucl. VII, 35)
22. $x \in \mathrm{N} . x < 41 : \supset . 41 - x + x^2 \in \mathrm{Np}.$
23. $\mathrm{M} . \mathrm{Np} : = \Lambda.$ (Eucl. IX, 20)
24. $n \in \mathrm{Np} . a \in \mathrm{N} . a - \text{ɑ}\, n : \supset . a^{n-1} - 1 \,\text{ɑ}\, n.$ (Fermat)

§ 8. Numerorum rationes.

Explicationes.

Si $p, q \in \mathrm{N}$, tunc $\frac{p}{q}$ legitur *ratio numeri* p *numero* q.

Signum R legitur *duorum numerorum ratio,* et indicat numeros rationales positivos.

Definitiones.

1. $m, p, q \,\epsilon\, N \,.\, \supset .\, m\, \frac{p}{q} = mp \,/\, q.$

2. $p, q, p', q' \,\epsilon\, N \,.\, \supset :: \frac{p}{q} = \frac{p'}{q'} \,.=\therefore x \,\epsilon\, N \,.\, x\, \frac{p}{q}, x\, \frac{p'}{q'} \,\epsilon\, N : \supset_x .\, x\, \frac{p}{q} =$ $x\, \frac{p'}{q'}.$

3. $R =:: [x\,\epsilon] \therefore p, q \,\epsilon\, N \,.\, \frac{p}{q} = x : - = \Lambda.$

4. $p \,\epsilon\, N \,.\, \supset .\, \frac{p}{1} = p.$

Theoremata.

5. $p, q, p', q' \,\epsilon\, N \,.\, \supset :: \frac{p}{q} = \frac{p'}{q'} \,.=.\, pq' = p'q.$ (EUCL. VII, 19)

Dem. $\text{Hp.} \frac{p}{q} = \frac{p'}{q'} : \supset \therefore qq', qq' \frac{p}{q}, qq' \frac{p'}{q'} \,\epsilon\, N \,.\, P2 \therefore \supset \therefore qq' \frac{p}{q} = qq' \frac{p'}{q'}$ $.\, qq' \frac{p}{q} = pq' \,.\, qq' \frac{p'}{q'} = p'q \therefore \supset \therefore pq' = p'q.$ (1)

$\text{Hp.}\, pq' = p'q \therefore \supset \therefore x \,\epsilon\, N \,.\, x \frac{p}{q}, x \frac{p'}{q'} \,\epsilon\, N : \supset_x : x pq' = x p'q : \supset$ $: \left(x \frac{p}{q}\right) qq' = \left(x \frac{p'}{q'}\right) qq' : \supset : x \frac{p}{q} = x \frac{p'}{q'}.$ (2)

$(1)(2) \,.\, \supset .\, \text{Th.}$

6. $m, p, q \,\epsilon\, N \,.\, \supset .\, \frac{p}{q} = \frac{mp}{mq}.$ (EUCL. VII, 17)

7. $p, q \,\epsilon\, N \,.\, m \,\epsilon\, N \,.\, m\, D\, p \,.\, m\, D\, q : \supset .\, \frac{p}{q} = \frac{p/m}{q/m}.$

8. $p, q, p', q' \,\epsilon\, N \,.\, p\, \pi\, q \,.\, p'\, \pi\, q' \,.\, \frac{p}{q} = \frac{p'}{q'} : \supset : p = p' \,.\, q = q'.$

9. $p, q, p', q' \,\epsilon\, N \,.\, p'\, \pi\, q' \,.\, \frac{p}{q} = \frac{p'}{q'} : \supset : p' / p = q' / q = M \ni D(p, q).$

10. $p, q, p', q' \,\epsilon\, N \,.\, \frac{p}{q} = \frac{p'}{q'} \,.\, p\, \pi\, q \,.\, q' < q : = \Lambda.$ (EUCL. VII, 21)

11. $p, q, p', q' \,\epsilon\, N : \supset : \frac{p}{q} = \frac{p'}{q'} \,.=.\, \frac{p}{p'} = \frac{q}{q'} \,.=.\, \frac{q}{p} = \frac{q'}{p'}.$ (EU. VII, 13)

12. $p, q \,\epsilon\, N \,.\, \supset :: [m\,\epsilon] : m \,\epsilon\, N \,.\, m \frac{p}{q} \,\epsilon\, N \therefore - = \Lambda.$

12'. $a \,\epsilon\, R \,.\, \supset :: [m\,\epsilon] : m \,\epsilon\, N \,.\, ma \,\epsilon\, N \therefore - = \Lambda.$

13. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset :: [(r, s, t)\,\epsilon] : r, s, t \,\epsilon\, \mathrm{N} \,.\, \frac{p}{q} = \frac{r}{t} \,.\, \frac{p'}{q'} = \frac{s}{t} \therefore - = \Lambda.$

13'. $a, b \,\epsilon\, \mathrm{R} \,.\supset :: [(r, s, t)\,\epsilon] : r, s, t \,\epsilon\, \mathrm{N} \,.\, a = \frac{r}{t} \,.\, b = \frac{s}{t} \therefore - = \Lambda.$

14. $a, b, c \,\epsilon\, \mathrm{R} \,.\supset :: [(m, n, p, q)\,\epsilon] : m, n, p, q \,\epsilon\, \mathrm{N} \,.\, a = \frac{m}{q} \,.\, b = \frac{n}{q} \,.\, c = \frac{p}{q} \therefore - = \Lambda.$

15. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \supset : a = b \,.=.\, p = q.$

16. $m \,\epsilon\, \mathrm{N} \,.\, a, b \,\epsilon\, \mathrm{R} \,.\, a = b \,.\, ma \,\epsilon\, \mathrm{N} : \supset \,.\, mb \,\epsilon\, \mathrm{N}.$

17. $a, b, c \,\epsilon\, \mathrm{R} \,.\supset \therefore a = a.$

$\supset \therefore a = b \,.=.\, b = a.$

$\supset \therefore a = b \,.\, b = c : \supset \,.\, a = c.$

18. $\mathrm{N} \supset \mathrm{R}.$

Definitiones.

19. $a, b \,\epsilon\, \mathrm{R} \,.\supset :: a < b \,.=\therefore x \,\epsilon\, \mathrm{N} \,.\, xa, xb \,\epsilon\, \mathrm{N} : \supset \,.\, xa < xb.$

20. $a, b \,\epsilon\, \mathrm{R} \,.\supset : b > a \,.=.\, a < b.$

Theoremata.

21. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \supset : a < b \,.=.\, p < q.$

22. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset : \frac{p}{q} < \frac{p'}{q'} \,.=.\, pq' < p'q.$

23. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{r}{p} \,.\, b = \frac{r}{q} : \supset : a < b \,.=.\, p > q.$

24. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, \frac{p}{q} < \frac{p'}{q'} : \supset \,.\, \frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}.$

25. $a \,\epsilon\, \mathrm{R} \,.\supset \therefore \mathrm{R} \,.\ni > a : - = \Lambda.$

26. $a \,\epsilon\, \mathrm{R} \,.\supset \therefore \mathrm{R} \,.\ni < a : - = \Lambda.$

27. $a, b \,\epsilon\, \mathrm{R} \,.\, a < b : \supset \therefore \mathrm{R} \,.\ni > a \,.\ni < b : - = \Lambda.$

28. $a, b \,\epsilon\, \mathrm{R} : \supset \therefore a < b \,.\, a = b : = \Lambda.$

$\supset \therefore a > b \,.\, a = b : = \Lambda.$

$\supset \therefore a < b \,.\, a > b : = \Lambda.$

$\supset \therefore a - < b \,.\, a - = b \,.\, a - > b : = \Lambda.$

29. $a, b, c \,\epsilon\, \mathrm{R} : \supset \therefore a < \cup = b \,.\, b < c : \supset : a < c.$

$\supset \therefore a < b \,.\, b < \cup = c : \supset : a < c.$

Definitiones.

30. $a, b \in \mathrm{R} . \supset . a + b = [c \epsilon] (c \in \mathrm{R} .\therefore x \in \mathrm{N} . x a, x b, xc \in \mathrm{N} : \supset_x . xa + xb = xc)$.

31. $a, b \in \mathrm{R} . \supset :: b - a = .\therefore [x \epsilon] (x \in \mathrm{R} . a + x = b)$.

32. $a, b \in \mathrm{R} . \supset . ab = [c \epsilon] (c \in \mathrm{R} .\therefore x \in \mathrm{N} . xa, (xa)b, xc \in \mathrm{N} : \supset_x . (xa) b = x c)$.

33. $a, b \in \mathrm{R} . \supset . b / a = [x \epsilon] (x \in \mathrm{R} . ax = b)$.

Theoremata.

34. $p, q, r \in \mathrm{N} . \supset . \frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$.

35. $a, b \in \mathrm{R} . \supset . a + b \in \mathrm{R}$.

36. $p, q, r \in \mathrm{N} . p < q : \supset . \frac{q}{r} - \frac{p}{r} = \frac{q-p}{r}$.

37. $a, b \in \mathrm{R} . a < b : \supset . b - a \in \mathrm{R}$.

38. $p, q, p', q' \in \mathrm{N} . \supset . \frac{p}{q} \frac{p'}{q'} = \frac{pp'}{qq'}$.

39. $a, b \in \mathrm{R} . \supset . ab \in \mathrm{R}$.

40. $p, q, p', q' \in \mathrm{N} . \supset . \frac{p}{q} / \frac{p'}{q'} = \frac{pq'}{p'q}$.

41. $a, b \in \mathrm{R} . \supset . b / a \in \mathrm{R}$.

42. $p, q \in \mathrm{N} . \supset . \frac{p}{q} = p / q$.

§ 9. Rationalium systemata. Irrationales.

Explicatio.

Si $a \in \mathrm{K\,R}$, signum $\mathrm{T}\,a$ legitur *terminus summus*, vel *limes summus classis a*. Supra hoc novum ens relationes ac operationes tantum definimus.

Definitiones.

1. $a \in \mathrm{KR} . x \in \mathrm{R} : \supset :: x < \mathrm{T}a . = .\therefore a . \ni > x : -= \Lambda$.

2. $a \in \mathrm{KR} . x \in \mathrm{R} : \supset ::: x = \mathrm{T}a . = : \cdot : a . \ni > x : = \Lambda :: u \in \mathrm{R} . u < x : \supset_u .\therefore a . \ni > u : -= \Lambda$.

3. $a \in \mathrm{KR} . x \in \mathrm{R} : \supset .\therefore x > \mathrm{T}a . = : x -< \mathrm{T}a . x -= \mathrm{T}a$.

Theorema.

4. $x \,\varepsilon\, \mathrm{R} . \supset :: x = .\therefore \mathrm{T} : \mathrm{R} . \ni < x.$

Explicatio.

Signum Q legitur *quantitas*, numerosque indicat reales positivos, rationales aut irrationales, 0 et ∞ exceptis.

Definitiones.

5. $\mathrm{Q} = [x \,\varepsilon] (a \,\varepsilon\, \mathrm{K\,R} : a - = \Lambda : \mathrm{R} \ni > \mathrm{T}\, a . - = \Lambda : \mathrm{T}\, a = x \therefore - = \Lambda).$

6. $a, b \,\varepsilon\, \mathrm{Q} . \supset :: a = b . = .\therefore \mathrm{R} . \ni < a : = : \mathrm{R} . \ni < b.$

7. $a, b \,\varepsilon\, \mathrm{Q} . \supset :: a < b . = .\therefore \mathrm{R} . \ni > a . \ni < b : - = \Lambda.$

8. $a, b \,\varepsilon\, \mathrm{Q} . \supset : b > a . = . a < b.$

Theoremata.

9. $a \,\varepsilon\, \mathrm{Q} . \supset .\therefore \mathrm{R} . \ni < a : - = \Lambda.$

10. $a \,\varepsilon\, \mathrm{Q} . \supset .\therefore \mathrm{R} . \ni > a : - = \Lambda.$

11. $\mathrm{R} \supset \mathrm{Q}.$

Subsistunt quoque propositiones quae a P 17, 28, 29 in § 8 obtinentur, si loco R legatur Q.

Definitiones.

12. $a, b \,\varepsilon\, \mathrm{Q} . \supset . a + b = \mathrm{T}\, [z \,\varepsilon] ([(x, y) \,\varepsilon] : x, y \,\varepsilon\, \mathrm{R} . x < a . y < b . x + y = z \therefore - = \Lambda).$

13. $a, b \,\varepsilon\, \mathrm{Q} . \supset . ab = \mathrm{T}\, [z \,\varepsilon] ([(x, y) \,\varepsilon] : x, y \,\varepsilon\, \mathrm{R} . x < a . y < b . xy = z \therefore - = \Lambda).$

Ut valeant hae definitiones, demonstrandum est subsistere propositiones 12 et 13, si $a, b \,\varepsilon\, \mathrm{R}$.

Subtractionem et divisionem ut operationes inversas additionis et multiplicationis definire licet, illarumque proprietates demonstrare.

§ 10. **Quantitatum systemata.**

Explicationes.

Si $a \,\varepsilon\, \mathrm{K\,Q}$, signa **I** a, **E** a, **L** a leguntur: *interior*, *exterior*, *limes classis* a.

Definitiones.

1. $a \in \mathrm{KQ} . \supset . \mathbf{I}\, a = \mathrm{Q}\, [x \in]\, ([(u, v) \in] :: u, v \in \mathrm{Q} \therefore u < x < v \therefore$
$\ni > u . \ni < v : \supset : a : \cdot : - = \Lambda)$.

2. $a \in \mathrm{KQ} . \supset . \mathbf{E}\, a = \mathbf{I}\, (- a)$.

3. $a \in \mathrm{KQ} . \supset . \mathbf{L}\, a = (- \mathbf{I}\, a)(- \mathbf{E}\, a)$.

Theoremata.

4. $a \in \mathrm{KQ} . x, u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a) : \supset . x \in \mathbf{I}\, a$.

5. $a \in \mathrm{KQ} . x \in \mathbf{I}\, a : \supset : [(u, v) \in]\, (u, v \in \mathrm{Q} \therefore u < x < v \therefore \ni > u .$
$\ni < v : \supset : a) - = \Lambda$.

Dem. P 1 = (P 4) (P 5).

6. $a \in \mathrm{KQ} . u, v \in \mathrm{Q} . (\ni > u . \ni < v : \supset a) \therefore \supset \therefore \ni > u . \ni < v : \supset \mathbf{I}\, a$.

Dem. P 6 = P 4.

7. $a \in \mathrm{KQ} . \supset . \mathbf{I}\, a \supset a$.

8. $a \in \mathrm{KQ} . \supset . \mathbf{II}\, a = \mathbf{I}\, a$.

Dem. Hp. $(\mathbf{I}\, a)\, [a]$ P 7 $: \supset . \mathbf{II}\, a \supset \mathbf{I}\, a$. (1)

Hp. $x, u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a)$. P 6 $: \supset : u, v$
$\in \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset \mathbf{I}\, a)$. (2)

Hp. $x \in \mathbf{I}\, a$. (2) $: \supset : x \in \mathbf{II}\, a$. (3)

Hp. (3) $: \supset : \mathbf{I}\, a \supset \mathbf{II}\, a$. (4)

Hp. (1) . (4) $: \supset :$ Ts. (Theor.)

9. $a, b \in \mathrm{KQ} . a \supset b : \supset . \mathbf{I}\, a \supset \mathbf{I}\, b$.

Dem. Hp. $x, u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a) : \supset \therefore \ni > u .$
$\ni < v : \supset b$. (1)

Hp. $x \in \mathbf{I}\, a : \supset : x \in \mathbf{I}\, b$. (Theor.)

10. $a, b \in \mathrm{KQ} : \supset : \mathbf{I}\, (ab) \supset \mathbf{I}\, a$.

Dem. $(ab, a)\, [a, b]$ P 9 . = . P 10.

11. $a, b \in \mathrm{KQ} . \supset . \mathbf{I}\, (ab) \supset (\mathbf{I}\, a)\, (\mathbf{I}\, b)$.

Dem. P 11 = : P 10 . $\cap$. $(b, a)\, [a, b]$ P 10.

12. $a, b \in \mathrm{KQ} . \supset . \mathbf{I}\, a \supset \mathbf{I}\, (a \cup b)$.

13. $a, b \in \mathrm{KQ} . \supset . \mathbf{I}\, a \cup \mathbf{I}\, b \supset \mathbf{I}\, (a \cup b)$.

14. $a, b \in \mathrm{KQ} . \supset . \mathbf{I}\, (ab) = (\mathbf{I} a)\, (\mathbf{I} b)$.

Dem. Hp. P 11 $: \supset . \mathbf{I}\, (ab) \supset (\mathbf{I} a)\, (\mathbf{I} b)$. (1)

Hp. $x \epsilon \mathrm{Q} . u, v \epsilon \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a) . u', v' \epsilon \mathrm{Q} . u' < x < v' . (\ni > u' . \ni < v' : \supset b) . u'' = \mathrm{M}(u \cup u') . v'' = \mathrm{W}(v, v') : \supset : u'', v'' \epsilon \mathrm{Q} . u'' < x < v'' . (\ni > u'' . \ni > v'' : \supset : ab)$. (2)

Hp. $x \epsilon \mathbf{I}a . x \epsilon \mathbf{I}b . (2) : \supset . x \epsilon \mathbf{I}(ab)$. (3)

Hp. $(3) : \supset : (\mathbf{I}a)(\mathbf{I}b) \supset \mathbf{I}(ab)$. (4)

Hp. $(1) . (4) : \supset$. Ts.

15. $a \epsilon \mathrm{KQ} . \supset . \mathbf{E}a \supset -a$.

Dem. P 15 $= (-a)[a]$ P 7.

16. $a \epsilon \mathrm{KQ} . \supset \therefore \mathbf{I}a . \mathbf{E}a : = \Lambda$.

Dem. Hp. P 7 . P 15 $: \supset \therefore \mathbf{I}a . \mathbf{E}a : \supset : a - a : = \Lambda$.

17. $a \epsilon \mathrm{KQ} . \supset . \mathbf{I}\,\mathbf{E}a = \mathbf{E}a$.

Dem. P 17 $= (-a)[a]$ P 8.

18. $a, b \epsilon \mathrm{KQ} . b \supset a : \supset . \mathbf{E}a \supset \mathbf{E}b$.

Dem. P 18 $= (-a, -b)[a, b]$ P 9.

19. $a, b \epsilon \mathrm{KQ} . \supset : \mathbf{E}a \cup \mathbf{E}b . \supset \mathbf{E}(ab)$.

20. $a, b \epsilon \mathrm{KQ} . \supset . \mathbf{E}(a \cup b) = (\mathbf{E}a)(\mathbf{E}b)$.

Dem. P 20 $= (-a, -b)[a, b]$ P 14.

21. $a \epsilon \mathrm{KQ} . \supset . \mathbf{L}(-a) = \mathbf{L}a$.

22. $a \epsilon \mathrm{KQ} . \supset \therefore \mathbf{I}a . \mathbf{L}a : = \Lambda$.

$\supset \therefore \mathbf{E}a . \mathbf{L}a : = \Lambda$.

$\supset \therefore -\mathbf{I}a . -\mathbf{E}a . -\mathbf{L}a : = \Lambda$.

Dem. P 22 = P 3.

23. $a \epsilon \mathrm{KQ} . \supset : a \supset . \mathbf{I}a \cup \mathbf{L}a$.

24. $a \epsilon \mathrm{KQ} . \supset . \mathbf{I}(a\mathbf{L}a) = \Lambda$.

Dem. Hp. P 14 . P 7 . P 22 $: \supset : \mathbf{I}(a\mathbf{L}a) . = . \mathbf{I}a\mathbf{I}\mathbf{L}a . \supset . \mathbf{I}a\mathbf{L}a . = . \Lambda$.

25. $a, b \epsilon \mathrm{KQ} . a \supset b : \supset : \mathbf{L}a . \supset . \mathbf{I}b \cup \mathbf{L}b$.

Dem. Hp. P 18 $: \supset : \mathbf{E}b \supset \mathbf{E}a : \supset : \mathbf{I}a \cup \mathbf{L}a . \supset . \mathbf{I}b \cup \mathbf{L}b : \supset$. Ts.

26. $a, b \epsilon \mathrm{KQ} . \supset : \mathbf{L}(ab) \supset . \mathbf{I}a\mathbf{L}b \cup \mathbf{I}b\mathbf{L}a \cup \mathbf{L}a\mathbf{L}b$.

Dem. Hp. $\supset : ab \supset a . ab \supset b$. P 25 $: \supset : \mathbf{L}(ab) \supset \mathbf{I}a \cup \mathbf{L}a . \mathbf{L}(ab) \supset \mathbf{I}b \cup \mathbf{L}b : \supset : \mathbf{L}(ab) \supset (\mathbf{I}a \cup \mathbf{L}a)(\mathbf{I}b \cup \mathbf{L}b) . \mathbf{L}(ab)(\mathbf{I}a)(\mathbf{I}b) = \mathbf{L}(ab)\mathbf{I}(ab) = \Lambda : \supset :$ Ts.

26'. $a, b \epsilon \mathrm{KQ} . \supset . \mathbf{L}(ab) \supset \mathbf{L}a \cup \mathbf{L}b$.

27. $a, b \epsilon \mathrm{KQ} . \supset : \mathbf{L}(a \cup b) = \mathbf{L}a\mathbf{E}b \cup \mathbf{L}b\mathbf{E}a \cup \mathbf{L}a\mathbf{L}b$.

Dem. P 27 $= (-a, -b)[a, b]$ P 26.

27′. $a, b \in KQ . \supset : \mathbf{L}(a \cup b) \supset \mathbf{L}a \cup \mathbf{L}b$.

28. $a \in KQ . \supset . \mathbf{L}\mathbf{I}a \supset \mathbf{L}a$.

Dem. Hp. P 7 : $\supset : \mathbf{I}a \supset a$. P 25 : $\supset : \mathbf{L}\mathbf{I}a \supset \mathbf{I}a \cup \mathbf{L}a$. (1)

Hp. P 8 . P 22 : $\supset . \mathbf{L}\mathbf{I}a\mathbf{I}a = \mathbf{L}\mathbf{I}a\mathbf{I}\mathbf{I}a = \Lambda$. (2)

(1) (2) . $\supset$. Theor.

28′. $a \in KQ . \supset . \mathbf{L}\mathbf{E}a \supset \mathbf{L}a$.

29. $a \in KQ . \supset . \mathbf{L}\mathbf{L}a \supset \mathbf{L}\mathbf{I}a \cup \mathbf{L}\mathbf{E}a$.

Dem. Hp. $\supset : \mathbf{L}\mathbf{L}a = \mathbf{L}(\mathbf{I}a \cup \mathbf{E}a)$. P 27′ : $\supset$. Ts.

29′. $a \in KQ . \supset . \mathbf{L}\mathbf{L}a \supset \mathbf{L}a$.

Dem. P 29 . P 28 . P 28′ : $\supset$. Theor.

30. $a \in KQ . \supset . \mathbf{L}a = \mathbf{I}\mathbf{L}a \cup \mathbf{L}\mathbf{L}a$.

Dem. Hp. P 23 : $\supset . \mathbf{L}a \supset \mathbf{I}\mathbf{L}a \cup \mathbf{L}\mathbf{L}a$. (1)

Hp. P 7 : $\supset . \mathbf{I}\mathbf{L}a \supset \mathbf{L}a$. (2)

Hp. P 29′ : $\supset . \mathbf{L}\mathbf{L}a \supset \mathbf{L}a$. (3)

(1) (2) (3) . $\supset$. Theor.

31. $a \in KQ . \supset . \mathbf{L}\mathbf{I}\mathbf{L}a \supset \mathbf{L}\mathbf{L}a$.

Dem. P 31 $=$ ($\mathbf{L}a$) $[a]$ P 28.

32. $a \in KQ . \supset . \mathbf{I}\mathbf{L}\mathbf{L}a = \Lambda$.

Dem. Hp. P 29′ : $\supset : \mathbf{L}\mathbf{L}a = \mathbf{L}a\mathbf{L}\mathbf{L}a$. ($\mathbf{L}a$) $[a]$ P 24 : $\supset$ Ts.

33. $a \in KQ . \supset : \mathbf{I}\mathbf{L}\mathbf{I}\mathbf{L}a = \Lambda$.

Dem. P 31 . P 32 : $\supset$. P 33.

34. $a \in KQ . \supset . \mathbf{L}\mathbf{L}\mathbf{L}a = \mathbf{L}\mathbf{L}a$.

Dem. ($\mathbf{L}a$) $[a]$ P 30 . P 32 : $\supset$. Theor.

35. $a, b \in KQ . \supset . \mathbf{I}a\mathbf{L}b \supset \mathbf{L}(ab)$.

Dem. Hp. P 14 : $\supset . \mathbf{I}a\mathbf{L}b\mathbf{I}(ab) = \mathbf{I}a\mathbf{I}b\mathbf{L}b = \Lambda$. (1)

Hp. P 2 . P 14 : $\supset . \mathbf{I}a\mathbf{L}b\mathbf{E}(ab) = \mathbf{I}a\mathbf{L}b\mathbf{I}(-a \cup -b) = \mathbf{I}(a - b)\mathbf{L}b = \mathbf{I}a\mathbf{E}b\mathbf{L}b = \Lambda$. (2)

(1) (2) $\supset$ Theor.

36. $a, b \in KQ . \supset . \mathbf{I}a\mathbf{L}b \cup \mathbf{I}b\mathbf{L}a \supset \mathbf{L}ab$. (Vide P 26)

Dem. P 36 $=$: P 35 . (b, a) $[a, b]$ P 35.

37. $a, b \in KQ . \supset . \mathbf{E}a\mathbf{L}b \cup \mathbf{E}b\mathbf{L}a \supset \mathbf{L}(a \cup b)$. (Vide P 27)

Dem. P 37 $=$ $(-a, -b)$ $[a, b]$ P 36.

38. $a, b \in KQ . \supset . \mathbf{I}(a \cup b) \supset \mathbf{I}a \cup \mathbf{I}b \cup \mathbf{L}a\mathbf{L}b$. (Vide P 13)

Dem. Hp. $\supset . \mathbf{I}(a \cup b) \supset (\mathbf{I}a \cup \mathbf{L}a \cup \mathbf{E}a)(\mathbf{I}b \cup \mathbf{L}b \cup \mathbf{E}b)$. (1)

Hp. P 20 . P 16 : $\supset . \mathbf{I}(a \cup b)\mathbf{E}a\mathbf{E}b = \mathbf{I}(a \cup b)\mathbf{E}(a \cup b) = \Lambda$. (2)

Hp. P 37 : $\supset$: $\mathrm{I}(a \cup b)(\mathrm{E}a\mathrm{L}b \cup \mathrm{E}b\mathrm{L}a) . \supset . \mathrm{I}(a \cup b)\mathrm{L}(a \cup b) .$
$= \Lambda .$ (3)

(1)(2)(3) . $\supset$. Theor.

38'. $a, b \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{E}(ab) \supset \mathrm{E}a \cup \mathrm{E}b \cup \mathrm{L}a\mathrm{L}b.$ (Vide P 19)

39. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{IL}a\mathrm{LI}a = \Lambda .$

Dem. Hp. P 36 : $\supset$: $\mathrm{IL}a\mathrm{LI}a \supset \mathrm{L}(\mathrm{L}a\mathrm{I}a) = \Lambda .$

40. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{LI}a \supset \mathrm{LL}a.$

Dem. Hp. P 28 . P 30 . P 39 : $\supset$ Theor.

40'. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{LE}a \supset \mathrm{LL}a.$

41. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{LL}a = \mathrm{LI}a \cup \mathrm{LE}a.$

Dem. P 29 . P 40 . P 40' : $\supset$. Theor.

42. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{ILI}a = \Lambda .$
$\supset . \mathrm{ILE}a = \Lambda .$
$\supset . \mathrm{LLI}a = \mathrm{LI}a.$
$\supset . \mathrm{LLE}a = \mathrm{LE}a.$

43. $a, b \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{I}(\mathrm{I}a \cup \mathrm{I}b) = \mathrm{I}a \cup \mathrm{I}b.$

Dem. Hp. P 7 : $\supset . \mathrm{I}(\mathrm{I}a \cup \mathrm{I}b) \supset \mathrm{I}a \cup \mathrm{I}b.$ (1)
Hp. P 8 . P 13 : $\supset$: $\mathrm{I}a \cup \mathrm{I}b . = . \mathrm{II}a \cup \mathrm{II}b . \supset . \mathrm{I}(\mathrm{I}a \cup \mathrm{I}b).$ (2)
(1)(2) $\supset$ Theor.

44. $a, b \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{I}(\mathrm{LL}a \cup \mathrm{LL}b) = \Lambda .$

Dem. Hp. P 38 . P 32 . P 34 : $\supset . \mathrm{I}(\mathrm{LL}a \cup \mathrm{LL}b) \supset \mathrm{LL}a\mathrm{LL}b \supset \mathrm{LL}a.$ (1)
Hp. (1) . P 8 : $\supset . \mathrm{I}(\mathrm{LL}a \cup \mathrm{LL}b) \supset \mathrm{ILL}a = \Lambda .$

45. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{I}(\mathrm{I}a \cup \mathrm{E}a) = \mathrm{I}a \cup \mathrm{E}a.$

Dem. P 8 . P 17 . $(-a)[b]$ P 43 : $\supset$. Theor.

45'. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{EL}a = \mathrm{I}a \cup \mathrm{E}a.$

46. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{EI}a = -(\mathrm{I}a \cup \mathrm{LI}a).$

46'. $a \,\epsilon\, \mathrm{KQ} . \supset . \mathrm{EE}a = -(\mathrm{E}a \cup \mathrm{LE}a).$

Hp. P 37 : $\supset$: $\mathbf{I}\,(a \cup b)\,(\mathbf{E}\,a\,\mathbf{L}\,b \cup \mathbf{E}\,b\,\mathbf{L}\,a)$. $\supset$. $\mathbf{I}\,(a \cup b)\,\mathbf{L}\,(a \cup b)$.
$= \Lambda$. (3)

(1) (2) (3) . $\supset$. Theor.

38'. $a, b \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{E}\,(ab) \supset \mathbf{E}\,a \cup \mathbf{E}\,b \cup \mathbf{L}\,a\,\mathbf{L}\,b$. (Vide P 19)

39. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{I\,L}\,a\,\mathbf{L\,I}\,a = \Lambda$.

Dem. Hp. P 36 : $\supset$: $\mathbf{I\,L}\,a\,\mathbf{L\,I}\,a \supset \mathbf{L}\,(\mathbf{L}\,a\,\mathbf{I}\,a) = \Lambda$.

40. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{L\,I}\,a \supset \mathbf{LL}\,a$.

Dem. Hp. P 28 . P 30 . P 39 : $\supset$ Theor.

40'. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{L\,E}\,a \supset \mathbf{LL}\,a$.

41. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{LL}\,a = \mathbf{L\,I}\,a \cup \mathbf{L\,E}\,a$.

Dem. P 29 . P 40 . P 40' : $\supset$. Theor.

42. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{I\,L\,I}\,a = \Lambda$.
$\supset$. $\mathbf{I\,L\,E}\,a = \Lambda$.
$\supset$. $\mathbf{LL\,I}\,a = \mathbf{L\,I}\,a$.
$\supset$. $\mathbf{LL\,E}\,a = \mathbf{L\,E}\,a$.

43. $a, b \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{I}\,(\mathbf{I}\,a \cup \mathbf{I}\,b) = \mathbf{I}\,a \cup \mathbf{I}\,b$.

Dem. Hp. P 7 : $\supset$. $\mathbf{I}\,(\mathbf{I}\,a \cup \mathbf{I}\,b) \supset \mathbf{I}\,a \cup \mathbf{I}\,b$. (1)
Hp. P 8 . P 13 : $\supset$: $\mathbf{I}\,a \cup \mathbf{I}\,b$. $=$. $\mathbf{II}\,a \cup \mathbf{II}\,b$. $\supset$. $\mathbf{I}\,(\mathbf{I}\,a \cup \mathbf{I}\,b)$. (2)
(1) (2) $\supset$ Theor.

44. $a, b \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{I}\,(\mathbf{LL}\,a \cup \mathbf{LL}\,b) = \Lambda$.

Dem. Hp. P 38 . P 32 . P 34 : $\supset$. $\mathbf{I}\,(\mathbf{LL}\,a \cup \mathbf{LL}\,b) \supset \mathbf{LL}\,a\,\mathbf{LL}\,b \supset \mathbf{LL}\,a$. (1)
Hp. (1) . P 8 : $\supset$. $\mathbf{I}\,(\mathbf{LL}\,a \cup \mathbf{LL}\,b) \supset \mathbf{I\,LL}\,a = \Lambda$.

45. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{I}\,(\mathbf{I}\,a \cup \mathbf{E}\,a) = \mathbf{I}\,a \cup \mathbf{E}\,a$.

Dem. P 8 . P 17 . $(-a)\,[b]$ P 43 : $\supset$. Theor.

45'. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{E\,L}\,a = \mathbf{I}\,a \cup \mathbf{E}\,a$.

46. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{E\,I}\,a = -(\mathbf{I}\,a \cup \mathbf{L\,I}\,a)$.

46'. $a \,\epsilon\, \mathrm{KQ}$. $\supset$. $\mathbf{EE}\,a = -(\mathbf{E}\,a \cup \mathbf{L\,E}\,a)$.